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Force-induced depinning of directed polymers

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Abstract

We present an approach to studying directed polymers in interaction with a defect line and subject to a force which pulls them away from the line. We consider in particular the case of inhomogeneous interactions. We first give a formula relating the free energy of these models to the free energy of the corresponding ones in which the force is switched off. We then show how to detect the presence of a re-entrant transition without fully solving the model. We discuss some models in detail and show that inhomogeneous interactions, e.g. disordered interactions, may induce the re-entrance phenomenon.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

1.1. Overview

Pulling polymers out of potential wells by applying a force f in the direction orthogonal to the pinning region has been considered at several instances (see e.g. [3, 16, 19, 20, 23] and references therein). The techniques employed essentially lead to exact (and mathematically rigorous) results for models with homogeneous interactions (see, in particular, [20, 23]), while in the inhomogeneous case, notably in the disordered case, the results are often based on arguments out of mathematical control (e.g., replica and renormalization group computations) or on annealed approximations that, in general, are not close to the behaviour of the quenched system. One of the interesting phenomena that has been pointed out, at least in some model systems, is the presence of a re-entrant transition (see in particular [20], but this issue is taken up in most of the references we have given). By this we mean that, for a fixed force f, the polymer is pulled out of the defect line at low temperature and at high temperature, but for

intermediate temperatures it is localized at the defect line, very much like if the force was not present. The references above are limited to theoretical work, but all of them are motivated by real experiments: the literature on real experiments is very rich, and here we single out [6] that deal with DNA unzipping and contains further references.

Our purpose is to point out that one can directly relate the free energy in the presence of a force to the free energy of the polymer with a free endpoint for a very large class of directed models and that the presence of a re-entrant transition is easily related to suitable asymptotic behaviour of the expressions appearing in this formula. As a consequence, we will give a simple necessary and sufficient condition for the re-entrance to take place for very general models (including disordered ones).

In order to be concrete we choose to deal with a precise and rather limited class of models, but the reader will realize that the method we present is very general. Extensions are considered explicitly in section 5. Moreover, the phenomenology of the restricted class of models we consider is already very rich.

We point out that we focus exclusively on cases in which the polymer interacts only at the defect line and other relevant cases, like the case in which bulk disorder is present (e.g. [10, 18]), are not considered here. On the other hand, it is straightforward to generalize the content of this paper to directed models of copolymers near a selective interface [11, 15].

1.2. A model

Let us consider a (p, q) walk, that is a random walk $S := \{S_n\}_{n=0,1,\dots}$ with $S_0 = 0$ and independent identically distributed (IID) increments $\{S_{n+1} - S_n\}_{n=0,1,\dots}$ taking values in $\{-1, 0, +1\}$, with $\mathbf{P}(S_{n+1} - S_n = 1) = \mathbf{P}(S_{n+1} - S_n = -1) = p/2$ and $\mathbf{P}(S_{n+1} - S_n = 0) = q$. We assume $p + q = 1$ and $p > 0$. Particularly relevant for what follows is the distribution of the first return time to zero, $\tau := \inf\{n : S_n = 0\}$: therefore we set $K(n) = \mathbf{P}(\tau = n)$, along with $\overline{K}(n) := \sum_{j>n} K(j)$. We point out that the Laplace transform of $K(\cdot)$, that is $\sum_n \exp(-bn)K(n)$, $b \geqslant 0$, can be written explicitly and $K(n)$ itself can be expressed in terms of $\mathbf{P}(S_k = 0)$, $k = 1, \dots, n$ (and therefore explicitly computed for every n) by using the standard renewal theory formula

$$\mathbf{P}(S_n = 0) = \delta_{n,0} + \sum_{j=1}^{n} K(j)\mathbf{P}(S_{n-j} = 0).$$

In particular one obtains that $n^{3/2}K(n)$ converges to $c_p := \sqrt{p/(2\pi)}$ ([12, appendix A]), if $p < 1$ (an analogous result holds for the $p = 1$ case, see below). Also note that $\overline{K}(0) = 1$, since S is recurrent.

Given a sequence $\omega := \{\omega_n\}_n$ of real numbers (the *charges*), we define the new measure $\mathbf{P}^{\beta,f}_{N,\omega}$ on the space of random walk paths by introducing a Boltzmann weight ($\beta \geqslant 0$, $f \geqslant 0$):

$$\frac{\mathrm{d}\mathbf{P}^{\beta,f}_{N,\omega}}{\mathrm{d}\mathbf{P}}(S) = \frac{1}{Z^{\beta,f}_{N,\omega}} \exp\left(\beta \sum_{n=1}^{N} \omega_n \mathbf{1}_{S_n=0} + \beta f S_N\right), \tag{1.1}$$

where $\mathbf{1}_{S_n=0}$ equals 1 if $S_n = 0$ and zero otherwise. For the reader unfamiliar with probabilistic notations, (1.1) means more explicitly the following. Let $s := \{s_n\}_{n=0,1,\dots,N}$ be a possible length N polymer configuration, so that in the absence of interaction with the defect line

$$\mathbf{P}(S_1 = s_1, \dots, S_N = s_N) = q^{\nu(s)}\,(p/2)^{N-\nu(s)}, \tag{1.2}$$

if $\nu(s) := \#\{1 \leqslant n \leqslant N : S_n = S_{n-1}\}$. Then, (1.1) says that in the presence of interaction

$$\mathbf{P}^{\beta,f}_{N,\omega}(S_1 = s_1, \dots, S_N = s_N) = \frac{q^{\nu(s)}(p/2)^{N-\nu(s)}}{Z^{\beta,f}_{N,\omega}} \exp\left(\beta \sum_{n=1}^{N} \omega_n \mathbf{1}_{s_n=0} + \beta f s_N\right)$$

where, of course,

$$Z_{N,\omega}^{\beta,f} = \sum_{s} q^{\nu(s)} (p/2)^{N-\nu(s)} \exp\left(\beta \sum_{n=1}^{N} \omega_n \mathbf{1}_{s_n=0} + \beta f s_N\right). \tag{1.3}$$

The Boltzmann average (for given charges ω and system size N) of an observable A will be denoted by $\mathbf{E}_{,\omega}^{\beta,f}(A)$.

We will consider two (wide) classes of charges, given below.

(1) ω is a periodic (deterministic) sequence, i.e. a sequence of numbers such that $\omega_{n+\mathrm{T}} = \omega_n$ for some positive integer T and every n. We denote by $\mathrm{T}(\omega)$ the minimal T with such a property, that is the *period*. If $\mathrm{T}(\omega) = 1$, then ω is homogeneous.
(2) ω is the realization of a sequence of random variables. For simplicity we consider only the case of IID variables. Therefore the law of ω is determined by the law of ω_1, but our analysis would go through in the much wider domain of stationary sequences of variables. We refer to this case as *disordered* or *quenched* and we denote by $\mathbb{P}$ the disorder distribution and by $\mathbb{E}$ the corresponding average.

The existence of the limit,

$$\lim_{N\to\infty} \frac{1}{N} \log Z_{N,\omega}^{\beta,0} =: \mathrm{F}(\beta), \tag{1.4}$$

is very well known. This follows from elementary super-additivity arguments in the periodic (and, of course, in the homogeneous) set-up (see e.g. [12, chapter 1]); in general, the limit depends on ω. In the disordered case, instead, one has to be a bit more careful; a precise statement is, for example, that if $\mathbb{E}|\omega_1| < \infty$ then the limit in (1.4) exists $\mathbb{P}$-almost surely (i.e. except for a set of disorder realization of probability zero). Moreover the right-hand side is in principle a random variable, but it turns out to be degenerate, i.e. it does not depend on ω. This property, usually referred to as *self-averaging*, is well understood in this context (see [12, chapter 4] for proofs and overview of the literature). We remark that

$$Z_{N,\omega}^{\beta,0} \geqslant \mathbf{P}[S_n \neq 0, \text{ for } n = 1, 2, \ldots, N] = \overline{K}(N) \geqslant cN^{-1/2}, \tag{1.5}$$

for some $c > 0$ (which follows from the definition of $\overline{K}(N)$ and from the fact that, as we mentioned above, $K(N)$ behaves like $N^{-3/2}$ for $N \to \infty$), so that $\mathrm{F}(\beta) \geqslant 0$ for every β. Much literature has been spent on this model, starting with the seminal works [7, 9], and about the fact that $\mathrm{F}(\beta) > 0$ corresponds to a localized regime, that is to the case in which the typical trajectories are tightly bound to the defect line (see [14] for rigorous results on the path properties in the localized phase and [12, chapter 7] for a full review of the literature). On the other hand, $\mathrm{F}(\beta) = 0$ corresponds to a delocalized regime. One should however distinguish between the critical point $\beta = \beta_c = \sup\{\beta : \mathrm{F}(\beta) = 0\}$ and the truly delocalized region where $\beta < \beta_c$. Both critical and delocalized regimes are well understood in the homogeneous/periodic case. The disordered case is instead mathematically much more elusive, but results have been obtained recently also in this case (see [12] and references therein, in particular [13, 25] for the disordered case), namely explicit estimates on how many visits (a non-extensive number!) are paid by the polymer to the defect line.

A relevant difference between the two classes of charges we consider is that in the first case $\mathrm{F}(\beta)$ is explicitly known. To be more precise in the periodic case, $\mathrm{F}(\beta)$ can be expressed in terms of the leading eigenvalue of a suitable $\mathrm{T}(\omega) \times \mathrm{T}(\omega)$ Perron–Frobenius matrix [4, 5] (see also appendix A.2); computing such an eigenvalue becomes harder and harder for larger periods, but the problem trivializes in the homogeneous case. In the special case of $T = 2$, the expressions for the free energy and other observable quantities have been derived, for example, in [21, 22]. In the disordered case, instead, only estimates on $\mathrm{F}(\beta)$ are known [1, 12].

We also point out that for every $y \in \mathbb{R}$, the limit

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbf{E}[\exp(yS_N);\, S_n \neq 0 \text{ for } n = 1, 2, \ldots, N] := \mathrm{G}(y), \tag{1.6}$$

exists. (Here we use the standard convention that if A is a set of trajectories and $g(S)$ an observable, then $\mathbf{E}(g(S);\, A) := \mathbf{E}(g(S)\mathbf{1}_A(S))$, where $\mathbf{1}_A(S)$ equals 1 if $S \in A$ and 0 otherwise.) Actually, the explicit value of $\mathrm{G}(\cdot)$ is easily computed by applying the reflection principle [12, appendix A] and one obtains $\mathrm{G}(y) = \log(p\cosh(y) + q)$.

For the model with the force we have:

Proposition 1.1. *For every β and f the limit, that we denote by $\mathrm{F}(\beta, f)$, of the sequence $\{(1/N)\log Z^{\beta,f}_{N,\omega}\}_N$ exists ($\mathbb{P}(\mathrm{d}\omega)$-almost surely (a.s.), in the disordered case). Moreover, we have the formula*

$$\mathrm{F}(\beta, f) = \max(\mathrm{F}(\beta), \mathrm{G}(\beta f)). \tag{1.7}$$

A line of non-analytic points of $\mathrm{F}(\cdot\,,\cdot)$ is evident: for every β we set $f_c(\beta) := \beta^{-1}\mathrm{G}^{-1}(\mathrm{F}(\beta)) \in [0, \infty)$. Therefore, for every β

$$\mathrm{F}(\beta, f) = \begin{cases} \mathrm{F}(\beta) & \text{if} \quad f \leqslant f_c(\beta), \\ \mathrm{G}(\beta f) & \text{if} \quad f \geqslant f_c(\beta). \end{cases} \tag{1.8}$$

It is quite easy to get convinced that this non-analyticity corresponds to a localization–delocalization transition; the most interesting case is when $\mathrm{F}(\beta) > 0$, otherwise the system is already delocalized at $f = 0$. By convexity, $\partial_f \mathrm{F}(\beta, f) > 0$ if $f > f_c(\beta)$ (and if $\partial_f \mathrm{F}(\beta, f)$ exists). This directly implies that $\lim_{N\to\infty} \mathbf{E}^{\beta,f}_{N,\omega} S_N/N = \partial_f \mathrm{F}(\beta, f) > 0$ For $f < f_c(\beta)$ instead $\lim_{N\to\infty} \mathbf{E}^{\beta,f}_{N,\omega} S_N/N = \partial_f \mathrm{F}(\beta, f) = 0$. These are distinctive marks respectively of localization and delocalization (for sharper results we refer to [24]).

It is also worthwhile to observe that formula (1.7) yields that $\partial_f \mathrm{F}(\beta, f)$ is discontinuous at $f = f_c(\beta)$ for $\beta > 0$. So the transition is of first order (as argued in much of the previous literature). The underlying mechanism has also been exploited in [2]. It is important to emphasize that this transition has in general nothing to do with the usual delocalization transition which the system, in the absence of pulling force, undergoes when the temperature or the average of the charges is varied. In particular, the latter transition is known rigorously to be always smooth (i.e. of order higher than the first) when disorder is present.

We now turn to the issue of the existence of a re-entrant transition. As mentioned above, this refers to the fact that for some fixed f the system undergoes (at least) two phase transitions as the temperature $T = 1/\beta$ is increased from zero to infinity. More precisely, two cases are observed: one can either observe a pattern of the type localized–delocalized–localized by increasing T, or the opposite one: delocalized–localized–delocalized.

Of course, re-entrance is equivalent to the non-monotonicity of f_c as a function of β, and a sufficient condition for it is that the difference $f_c(\infty) - f_c(0)$ has the opposite sign as $\partial_\beta f_c(\beta)|_{\beta=0}$. As we show in the following, these quantities are related to the asymptotic behaviour of $\mathrm{F}(\beta)$ for $\beta \to 0$ and $\beta \to \infty$, which in many cases (including quenched disordered situations) can be easily computed without fully solving the model.

We wish to emphasize that while, in principle, the occurrence of

$$\partial_\beta f_c(\beta)|_{\beta=0}[f_c(\infty) - f_c(0)] < 0, \tag{1.9}$$

is just a sufficient condition for re-entrance, it turns out numerically in the cases we have checked (cf figures 1 and 2) that when (1.9) is not verified the critical force is monotone in β and re-entrance is absent.

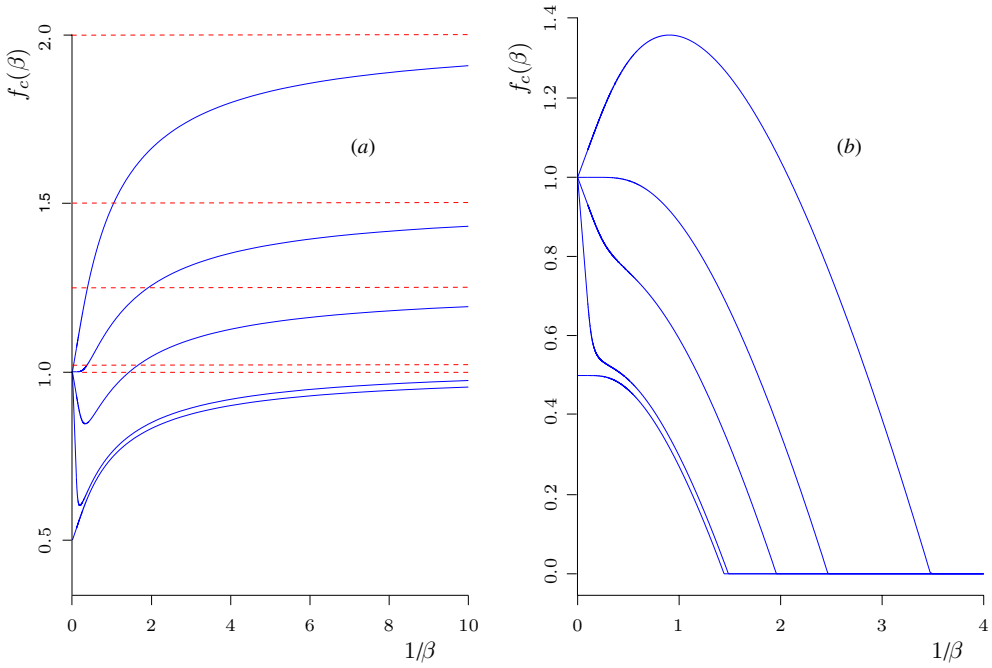

Figure 1. Critical force f_c as a function of $1/\beta$, for the homogeneous model. (*a*) Without a hard wall. The curves correspond to $q = 1/2, 1/3, 1/5, 1/50$ and 0, from top to bottom. The slope at the origin is positive if $\log q > \log(p/2)$, in agreement with (2.2) and therefore re-entrance is present if and only if $0 < q < 1/3$. The critical force $f_c(\beta)$ is computed by using (1.7), where $\text{F}(\beta)$ is the solution of $\sum_n K(n)\exp(-\text{F}(\beta)n) = \exp(-\beta)$ [12, proposition 1.1]. Note that the transition $p \to 1$ to the simple random walk case is singular, that is $f_c(\infty) = 1$ for every $p < 1$, but $f_c(\infty) = 1/2$ for $p = 1$ (compare (2.2) and (2.3)). (*b*) With a hard wall (the values of q are the same as in (*a*)). Re-entrance is present if and only if $q > 1/3$. In this case, $\text{F}(\beta)$ is given by the solution of $\sum_n K^+(n)\exp(-\text{F}(\beta)n) = \exp(-\beta)$, where $K^+(n) = K(n)/2$ if $n \geqslant 2$ and $K^+(1) = K(1)$.

2. The homogeneous case

This section has two aims:

(1) generalizing the results of the previous literature. For example in [23] only the cases $q = 1/3$ and $q = 0$ are considered and no re-entrance is observed, while for different values of q re-entrance does appear;
(2) providing the general scheme that we also follow in the following sections, even if the models of this section are exactly solvable (but we will not solve them).

First of all note that if $\omega_n \equiv -c < 0$, the polymer is delocalized already in the absence of the force and $f_c(\beta) = 0$ for every $\beta \geqslant 0$, an uninteresting situation. Therefore, in the homogeneous case, we will assume that $\omega_n \equiv c > 0$ and, without loss of generality, $c = 1$.

The behaviours of the critical force for $\beta \searrow 0$ and for $\beta \to \infty$ are easily obtained:

Proposition 2.1. *For the homogeneous model, i.e. (1.1) and* $\omega_n \equiv 1$, *with* $p < 1$ *one has*

$$\lim_{\beta \searrow 0} f_c(\beta) = \frac{1}{p}, \tag{2.1}$$

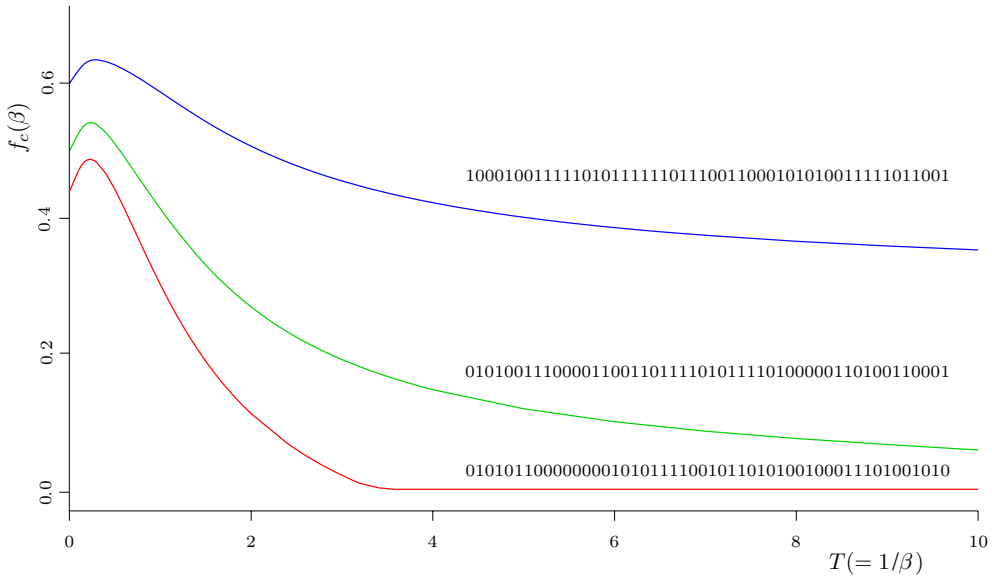

Figure 2. Critical force f_c as a function of $1/\beta$, in the case of periodic charge distribution and $q = 1/3$. In the three cases $\tau(\omega) = 50$ and the charges have been chosen at random, i.e. each sequence $\{\omega_n\}_{n=1,\dots,50}$, $\omega_n \in \{-1, 1\}$, is equiprobable (the explicit sequences $\{(\omega_n+1)/2\}_{n=1,\dots,50}$ are given next to the corresponding curves): for the top curve $|\{n : \omega_n = +1\}| = 30$, while for the bottom one $|\{n : \omega_n = +1\}| = 22$. The intermediate curve corresponds instead to the case in which there are as many positive charges as negative, and in fact the curve approaches 0 for large temperatures. As explained in the text, these graphs give also strong hints on the behaviour of disordered systems.

and

$$f_c(\beta) \overset{\beta\to\infty}{=} 1 + \frac{1}{\beta}(\log q - \log(p/2)) + o\left(\frac{1}{\beta}\right). \tag{2.2}$$

In particular, one has re-entrance if and only if $2/3 < p < 1$. *As for* $p = 1$, *one has* $f_c(\beta) \to 1$ *for* $\beta \to 0$ *and*

$$f_c(\beta) \overset{\beta\to\infty}{=} \frac{1}{2} + \frac{1}{2\beta}\log 2 + o\left(\frac{1}{\beta}\right). \tag{2.3}$$

Re-entrant behaviour is not observed in this case.

(Recall here that, in standard notation, if $x \to 0, o(x)$ denotes any quantity such that $o(x)/x \to 0$ for $x \to 0$.) See also figure 1(*a*), were we plot $f_c(\beta)$ as a function of $1/\beta$, for different choices of q.

Proof of proposition 2.1. Consider first the case $p < 1$. Since $K(n) \sim c_p n^{-3/2}, c_p = \sqrt{p/(2\pi)}$, for $\beta \searrow 0$ one has [12, Theorem 2.1]

$$\mathrm{F}(\beta) \sim \frac{\beta^2}{(2c_p\Gamma(1/2))^2}. \tag{2.4}$$

(To avoid any confusion, let us recall here that the standard notation $f(x) \sim g(x)$ for $x \to 0$ (or for $x \to \infty$) simply means that $f(x)/g(x)$ tends to 1 for $x \to 0$ (or for $x \to \infty$,

respectively).) This, together with a Taylor expansion of $G(y)$ around $y = 0$, immediately gives (2.1).

The large β behaviour can also be easily captured without explicit computations: for $p < 1$, $F(\beta) = \beta + \log K(1) + o(1)$ as $\beta \to \infty$, and $K(1) = q = 1 - p$, where $o(1)$ simply denotes a quantity which tends to zero for $\beta \to \infty$. This corresponds simply to the fact that for β large, the dominant trajectories are those such that $|\{n : S_n \neq 0\}| \ll N$. Since $G(y) = y + \log(p/2) + o(1)$ as $y \to \infty$, one finds (2.2).

For $p = 1$, (2.4) is modified in this case into $F(\beta) \sim \beta^2/2$ (this follows from [12, Theorem 2.1] plus the fact that, for the simple random walk, $K(2n) \sim n^{-3/2}\sqrt{1/(4\pi)}$ [8, chapter III]) and $K(2n+1) = 0$). As a consequence, $f_c(\beta) \to 1$ for $\beta \to 0$. As for the $\beta \to \infty$ behaviour, note that $F(\beta) = \beta/2 + (\log K(2))/2 + o(1)$ and $K(2) = 1/2$, so that (2.3) follows.

The statements about re-entrance easily follow from (2.1)–(2.3). □

2.1. The model with hard-wall repulsion

We conclude this section by showing that the phase diagram and the re-entrance phenomenon are strongly model dependent. To this purpose, we modify the model by inserting a hard-wall condition (like in [23]), which corresponds to inserting on the right-hand side of (1.1) the indicator function of the event $\{S : S_n \geqslant 0 \text{ for } n = 1, \ldots, N\}$. Of course, the partition function and the free energy will be in general modified (in particular, the sum in (1.3) will be restricted to configurations satisfying $s_i \geqslant 0, i = 1, \ldots, N$). We consider for conciseness only the case $p < 1$.

Proposition 2.2. *For the homogeneous model with hard-wall repulsion and $p < 1$, there exists $\beta_0 > 0$ such that $f_c(\beta) = 0$ for $\beta < \beta_0$. Moreover, the large β behaviour of $f_c(\beta)$ is still given by (2.2). Re-entrance takes place for $p < 2/3$.*

Proof of proposition 2.2. In the presence of the hard wall, for β sufficiently small but finite, one has $F(\beta) = 0$ (cf for instance [12, Section 1.2]) so that $f_c(\beta) = 0$. This is rather intuitive: even in the absence of the pulling force, the entropic repulsion effect provided by the wall is enough to delocalize the polymer. As for the $\beta \to \infty$ limit, since the dominant trajectory $S_n \equiv 0$ is allowed by the hard-wall constraint, one again finds $F(\beta) = \beta + \log K(1) + o(1)$, and equation (2.2). Occurrence of re-entrance for $p < 2/3$ immediately follows from the small- and large-β behaviour of the critical force. □

Observe that for the model with hard-wall repulsion, the situation is somewhat reversed with respect to the previous case: one has re-entrance for $p < 2/3$ and no re-entrance for $2/3 < p < 1$ (see also figure 1(*b*)).

3. The periodic case

We start by recalling that the solution of the periodic case can be reduced to a finite dimensional, in fact $T(\omega)$–dimensional, problem [4, 5, 12]; by solving numerically this finite dimensional problem we have drawn the curves in figure 2. However, from the $T(\omega)$–dimensional problem one can extract many analytic features too. Here we concentrate on small and large β behaviour, since they suffice to highlight part of the variety of observed phenomena. We collect and discuss here the results; see appendix A.2 for the proofs. We will assume for simplicity that charges ω_n take only the values ± 1. Just a bit of notation for the following result (for the sake of conciseness, we restrict our attention to the case without hard-wall

repulsion). Set $\ell_0 := \min\{n : \omega_n = +1\}$ (we are implicitly assuming that ω_n is not equal to -1 for all n) and $\ell_{i+1} := \inf\{n > \ell_i : \omega_n = +1\}$. Set also $\iota(\omega) := \min\{i : \ell_i > \mathrm{T}(\omega)\}$.

Proposition 3.1. *In the periodic case, we have the small temperature behaviour*

$$f_c(\beta) \overset{\beta\to\infty}{=} \frac{1}{\mathrm{T}(\omega)}\sum_{n=1}^{\mathrm{T}(\omega)}(2\omega_n - 1) + \frac{1}{\beta}\left(\frac{1}{\mathrm{T}(\omega)}\sum_{i=1}^{\iota(\omega)}\log K(\ell_i - \ell_{i-1}) - \log(p/2)\right) + o(1/\beta), \tag{3.1}$$

and at high temperatures ($\beta \searrow 0$)

$$f_c(\beta) = \begin{cases} \frac{1}{p\mathrm{T}(\omega)}\sum_{n=1}^{\mathrm{T}(\omega)}\omega_n + O(\beta) & \text{if} \quad \sum_{n=1}^{\mathrm{T}(\omega)}\omega_n > 0, \\ \frac{1}{2p}\beta + o(\beta) & \text{if} \quad \sum_{n=1}^{\mathrm{T}(\omega)}\omega_n = 0, \\ 0 & \text{if} \quad \sum_{n=1}^{\mathrm{T}(\omega)}\omega_n < 0 \quad \text{and} \quad \beta \leqslant \beta_0, \end{cases} \tag{3.2}$$

where $\beta_0 := \sup\{\beta : \mathrm{F}(\beta) = 0\}$, *and* $\beta_0 \in (0, \infty)$ *if* $\sum_{n=1}^{\mathrm{T}(\omega)}\omega_n < 0$.

We recall that $O(\beta)$ denotes any quantity whose absolute value is bounded above by a constant times β, for β sufficiently small.

Of course such a result is sufficient in order to check our sufficient condition (1.9) for the occurrence of a re-entrant transition, for a given periodic charge sequence ω. On the other hand, a full characterization of the critical curve in general requires some numerical computations (but we stress once again that they are just finite dimensional computations). As the example depicted in figure 2 shows, the inhomogeneous character of the charges may induce a re-entrant transition for example if $q = 1/3$, a value for which the homogeneous system shows no re-entrance, cf proposition 2.1 (but of course it can induce it also for $q < 1/3$).

An important feature resulting from proposition 3.1 (but also from figure 2) is the strong dependence on ω. In particular, the sign of the mean over a period leads to drastically different behaviours for large temperatures. It should be however pointed out that if $\omega_1, \ldots, \omega_{\mathrm{T}}$ are sampled in a IID fashion, like in the quenched disordered case, then in the limit of $\mathrm{T} \to \infty$ the free energy of the periodic model converges $\mathbb{P}(\mathrm{d}\omega)$–a.s. towards the free energy of the disordered model [12, Theorem 4.5]. One then directly sees that such a result implies the convergence of the critical force of the model with period T to the critical force of the corresponding disordered model. In figure 2, we have plotted a case in which $\sum_{n=1}^{\mathrm{T}}\omega_n = 0$ and two cases deviating above and below the mean by more than one standard deviation.

4. The disordered case

Let us now have a look at the disordered model. For the sake of conciseness, we consider only the case $p < 1$ and $\mathbb{P}(\omega_1 = +1) = \mathbb{P}(\omega_1 = -1) = 1/2$.

It is known that in this case the model with $f = 0$ and without hard-wall repulsion is localized for every $\beta > 0$ (see [12, chapter 5]). The behaviour of the free energy is not fully under control, but one can prove the following.

Proposition 4.1. *For the disordered model,* $f_c(\beta) \to 0$ *for* $\beta \to 0$. *As for large* β, *one has*

$$f_c(\beta) \overset{\beta\to\infty}{=} \frac{1}{2} + \frac{1}{\beta}\left(\frac{1}{2}\sum_{j=1}^{\infty}2^{-j}\log K(j) - \log(p/2)\right) + o\left(\frac{1}{\beta}\right) \tag{4.1}$$

so that re-entrance is observed as soon as

$$\frac{1}{2}\sum_{j=1}^{\infty} 2^{-j}\log K(j) > \log(p/2), \tag{4.2}$$

that is for $q > 0.1994\ldots$.

Proof of proposition 4.1. For $\beta \to 0$, the annealed bound

$$\mathrm{F}(\beta) \leqslant \lim_{N\to\infty} \frac{1}{N}\log \mathbb{E} Z_{N,\omega}^{\beta,0} \tag{4.3}$$

is already sufficient to prove that $f_c(\beta) \to 0$. In fact, the right-hand side is just the free energy of the homogeneous model where β is replaced by $\log(\cosh\beta)$, and therefore (recall (2.4) and discussion thereafter) it behaves like $\beta^4/(8p)$ for β small. This immediately implies that $f_c(\beta) \to 0$ for $\beta \to 0$.

Remark 4.2. In [1] it is actually proven that $\mathrm{F}(\beta) \overset{\beta\searrow 0}{\sim} \beta^4/(8p)$, so that $f_c(\beta) \overset{\beta\searrow 0}{\sim} \beta/(2\sqrt{p})$.

As for large β, we need the following.

Lemma 4.3. *For every* $\varepsilon > 0$ *there exists* β_ε *such that for* $\beta \geqslant \beta_\varepsilon$, *we have*

$$\left|\mathrm{F}(\beta) - \frac{\beta}{2} - \frac{1}{2}\sum_{j=1}^{\infty} 2^{-j}\log K(j)\right| \leqslant \varepsilon. \tag{4.4}$$

Note that (4.1) is a direct consequence of lemma 4.3, and the proof of the latter can be found in appendix A.1. Here, we just note that (4.4) shows how the situation is different from the homogeneous case, where one has rather $\mathrm{F}(\beta) = \beta + \log K(1) + o(1)$. In fact also in the disordered situation the dominant configurations are those for which $S_n = 0$ for every n such that $\omega_n = +1$ and $S_n \neq 0$ otherwise. However, now the average density of +1 charges is $1/2$ (whence the leading term $\beta/2$) and the distance between two successive positive charges is random and can take any value $j = 1, 2, \ldots$ with geometric probability 2^{-j}.

As explained in section 3, we expect the graph of $f_c(\cdot)$ to be close to the intermediate curve in figure 2.

4.1. The model with repulsion

Like in the homogeneous case, if we add the hard-wall repulsion then for β sufficiently small $\mathrm{F}(\beta) = f_c(\beta) = 0$ (see, for instance, [12, proposition 5.1]). Moreover, a look at the proof of (4.4) shows that lemma 4.3 still holds in this case, provided that $K(n)$ is replaced by $K^+(n)$ defined as $K^+(1) := K(1)$ and $K^+(n) := K(n)/2$ for $n \geqslant 2$. This is just related to the fact that, if $n \geqslant 2$, of all the possible trajectories $\{S_0, \ldots, S_n\}$ of the (p, q) walk satisfying $S_0 = S_n = 0, S_i \neq 0$ for $1 \leqslant i < n$, only half survive the introduction of the hard-wall constraint. Therefore, (4.1) still holds with $K(.)$ replaced by $K^+(.)$ and a sufficient condition for having a re-entrant transition is

$$\frac{1}{2}\sum_{j=1}^{\infty} 2^{-j}\log K^+(j) > \log(p/2). \tag{4.5}$$

Numerically, this corresponds to $q > 0.2838\ldots$.

5. Generalizations

The procedure presented in this paper to study the phase diagram and the occurrence of a re-entrant phase transition applies well beyond the case of the (p, q) walks. We have seen, in fact, that all what one needs to know is the asymptotic behaviour of $\mathrm{G}(y)$ for small and large y and the asymptotic behaviour of $\mathrm{F}(\beta)$ for small and large β.

Just to show an example of generalization, consider the case of a directed polymer in (2+1) dimensions with homogeneous ($\omega_n \equiv 1$) pinning attraction to a defect line. In particular, we will assume that $S_n \equiv \left(S_n^{(1)}, S_n^{(2)}\right) \in \mathbb{Z}^2$ and $\{S_n\}_{n=0,1,\dots}$ is the two-dimensional simple random walk where S_{n+1} is chosen uniformly among the four neighbours of S_n. In this section, it will be understood that $K(n)$ is the probability that the first return to zero of the two-dimensional random walk occurs at time n (in the previous sections, the same symbol was used for the analogous quantity referring to the (p, q) random walk).

Again, the Boltzmann weight is defined as in (1.1), with the only difference that the term $f S_N$ is replaced by, say, $f S_N^{(1)}$. This corresponds to assuming that the force is pulling in the direction '1' (orthogonal to the defect line). Then, proposition 1.1 still holds, provided that in the definition of $\mathrm{G}(y)$ one puts $y S_N^{(1)}$ instead of $y S_N$.

The analogue of proposition 2.1 is the following.

Proposition 5.1. *For the homogeneous* $(2 + 1)$*-dimensional model one has* $f_c(\beta) \to 0$ *for* $\beta \to 0$ *and*

$$f_c(\beta) \overset{\beta\to\infty}{=} \frac{1}{2} + \frac{1}{\beta}\log 2 + o\left(\frac{1}{\beta}\right). \tag{5.1}$$

Re-entrance does take place.

Proof of proposition 5.1. It is not difficult to prove that for the two-dimensional simple random walk,

$$\mathrm{G}(y) = \log(\cosh(y) + 1) - \log 2. \tag{5.2}$$

Indeed, first of all one has

$$\begin{aligned}\mathbf{E}\left[\exp\left(yS_N^{(1)}\right); S_n \neq 0 \text{ for } n = 1, 2, \dots, N\right] &\leqslant \mathbf{E}\left[\exp\left(yS_N^{(1)}\right)\right] = \mathbf{E}\left[\exp\left(yS_1^{(1)}\right)\right]^N \\ &= (1/2 + (1/2)\cosh y)^N \end{aligned} \tag{5.3}$$

for every N, so that

$$\mathrm{G}(y) \leqslant \log(\cosh(y) + 1) - \log 2. \tag{5.4}$$

To get the complementary lower bound, note that

$$\mathbf{E}\left[\exp\left(yS_N^{(1)}\right); S_n \neq 0 \text{ for } n = 1, \dots, N\right] \geqslant \mathbf{E}\left[\exp\left(yS_N^{(1)}\right); S_n^{(1)} \neq 0 \text{ for } n = 1, \dots, N\right]. \tag{5.5}$$

The quantity averaging on the right-hand side of (5.5) involves only $\left\{S_n^{(1)}\right\}_{n\geqslant 0}$, which is just a (p, q) walk with $p = 1/2$. Therefore, from the knowledge of $\mathrm{G}(y)$ for the (p, q) walk (cf section 1.2), one obtains $\mathrm{G}(y) \geqslant \log(1/2 + (1/2)\cosh(y))$ which concludes the proof of (5.2).

Next, it is known [17] that $K(2n) \sim c/(n(\log n)^2)$ with $c > 0$ while $K(2n + 1) = 0$. As a consequence (cf [12, Theorem 2.1]), $\mathrm{F}(\beta)$ vanishes for $\beta \to 0$ faster than any power of β. This, together with the expansion $\mathrm{G}(y) = y^2/4 + o(y^2)$ for $y \to 0$, implies that $f_c(\beta) \to 0$. As for the large-β behaviour, in analogy with the discussion in section 2, a look at the dominant trajectories gives $\mathrm{F}(\beta) = \beta/2 + (\log K(2))/2 + o(1/\beta)$, where now $K(2) = 1/4$. Since $\mathrm{G}(y) = y - 2\log 2 + o(1)$ for $y \to \infty$, (5.1) follows from proposition 1.1. □

The disordered $(2+1)$-dimensional model where ω_n are IID symmetric random variables $\omega_n = \pm 1$ can also be analysed in our framework, with the following result.

Proposition 5.2. *For the disordered* $(2+1)$*-dimensional pinning model,* $f_c(\beta) \to 0$ *for* $\beta \to 0$ *and*

$$f_c(\beta) \overset{\beta\to\infty}{=} \frac{1}{4} + \frac{1}{\beta}\left[2\log 2 + \frac{1}{4}\sum_{j\geqslant 1} 2^{-2j}\log K(2j)\right] + o\left(\frac{1}{\beta}\right). \tag{5.6}$$

Re-entrance is observed.

The proof of proposition 5.2 is essentially identical to that of proposition 4.1. The reason why the factors $1/2$ of (4.1) are replaced by $1/4$ is just that our two-dimensional walk can touch the defect line only for n even, and therefore for large β dominant trajectories will touch half of the total positive charges, i.e. approximately $N/4$ of them. The occurrence of re-entrance follows from a numerical evaluation of the constant multiplying $1/\beta$ in (5.6), which turns out to be positive (and equal to $1.2427\ldots$).

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A. Proofs and technical estimates

A.1. Proof of proposition 1.1 and lemma 4.3

Proof of proposition 1.1. Decompose the trajectory according to the location of the last visit of S to the origin before N (including N), getting thus by the Markov property of S

$$Z^{\beta,f}_{N,\omega} = \sum_{m=0}^{N} \mathbf{E}\left[\exp\left(\beta\sum_{n=1}^{m}\omega_n \mathbf{1}_{S_n=0}\right); S_m = 0\right] \times \mathbf{E}[\exp(\beta f S_{N-m}); S_k \neq 0, k = 1,\ldots,N-m]. \tag{A.1}$$

The right-hand side has the form $\sum_{m=0}^{N} a_m(\omega)b_{N-m}$, with $a_0(\omega) = b_0 = 1, a_m(\omega) = \exp((\mathrm{F}(\beta) + o(1))m)$ for $m \to \infty$, $\mathbb{P}(\mathrm{d}\omega)$-a.s. (the presence of the indicator function of the event $S_m = 0$ is irrelevant, see e.g. [12, remark 1.2], and $b_m = \exp((\mathrm{G}(\beta f) + o(1))m)$, again for $m \to \infty$. At this point, we remark that $Z^{\beta,f}_{N,\omega} \geqslant \max(a_N(\omega), b_N)$ and therefore $\liminf_{N\to\infty}(1/N)\log Z^{\beta,f}_{N,\omega}$ is bounded below by $\max(\mathrm{F}(\beta), \mathrm{G}(\beta f))$. For the opposite inequality we note that for every $\varepsilon > 0$, there exists $A(\omega)$ such that $a_n(\omega) \leqslant A(\omega)\exp((\mathrm{F}(\beta) + \varepsilon)n)$ for every n. Analogously, $b_n(\omega) \leqslant B\exp((\mathrm{G}(\beta) + \varepsilon)n)$ for some constant B and every n. Therefore,

$$Z^{\beta,f}_{N,\omega} \leqslant A(\omega)B\sum_{m=0}^{N}\exp(\mathrm{F}(\beta)m + \mathrm{G}(\beta f)(N-m) + \varepsilon N), \tag{A.2}$$

so that

$$Z^{\beta,f}_{N,\omega} \leqslant (N+1)A(\omega)B\exp(\max(\mathrm{F}(\beta), \mathrm{G}(\beta f))N + \varepsilon N), \tag{A.3}$$

for every N and, since $\varepsilon > 0$ can be chosen arbitrarily small, the proof is complete. □

Proof of lemma 4.3. We set

$$Q := \frac{1}{2}\sum_{j=1}^{\infty} 2^{-j}\log K(j), \tag{A.4}$$

and we separate the proof in lower and upper bounds.

For the lower bound, we select the S trajectories hitting 0 if and only if the charge is +1 on that site. This yields

$$Z^{\beta,0}_{N,\omega} \geqslant \left(\prod_{j=1}^{\mathcal{N}_N(\omega)} \exp(\beta)K(\ell_j)\right)\overline{K}\left(N - \sum_{j=1}^{\mathcal{N}_N(\omega)}\ell_j\right), \tag{A.5}$$

where $\ell_1 := \inf\{n > 0 : \omega_n = 1\}$, $\ell_{k+1} := \inf\left\{n > 0 : \omega_{n+\sum_{j=1}^{k}\ell_j} = 1\right\}$ and $\mathcal{N}_N(\omega) := \max\left\{k : \sum_{j=1}^{k}\ell_j \leqslant N\right\}$. Note that $\{\ell_j\}_j$ is an IID sequence of geometric random variables of parameter $1/2$. By taking the logarithm and dividing by N both sides in (A.5), in the limit as $N \to \infty$ we get

$$\mathrm{F}(\beta) \geqslant \beta \limsup_{N\to\infty}\frac{1}{N}\mathcal{N}_N(\omega) + \limsup_{N\to\infty}\frac{1}{N}\sum_{j=1}^{\mathcal{N}_N(\omega)}\log K(\ell_j), \tag{A.6}$$

where we have used the fact that $\overline{K}\left(N - \sum_{j=1}^{\mathcal{N}_N(\omega)}\ell_j\right) \geqslant \overline{K}(N)$ and $(\log\overline{K}(N))/N \to 0$ as $N \to \infty$. By the classical renewal theorem $\mathcal{N}_N(\omega)/N$ actually converges $\mathbb{P}(\mathrm{d}\omega)$-a.s. to $1/\mathbb{E}[\ell_1] = 1/2$ and, in turn, by the strong law of large numbers also the second superior limit on the right-hand side of (A.6) is an almost sure limit and it is equal to $\mathbb{E}[\log K(\ell_1)]/\mathbb{E}[\ell_1]$, which coincides with Q. This concludes the proof of the lower bound.

For the upper bound, we define $A_{N,\omega} := \{n : \omega_n = +1\} \cap [1, N]$ and for every $\varepsilon > 0$ and any realization of ω, we define the set of trajectories

$$\Omega_{N,\omega,\varepsilon} := \{S : |\{1 \leqslant n \leqslant N : S_n = 0\} \triangle A_{N,\omega}| \leqslant \varepsilon|A_{N,\omega}|\}, \tag{A.7}$$

where $\triangle$ denotes the symmetric difference of sets. Of course, by the law of large numbers $|A_{N,\omega}|/N \overset{N\to\infty}{\longrightarrow} 1/2$, $\mathbb{P}(\mathrm{d}\omega)$-a.s. Therefore for the partition function restricted to $\Omega_{N,\omega,\varepsilon}$, we have

$$Z^{\beta,0}_{N,\omega}\left(\Omega^{\complement}_{N,\omega,\varepsilon}\right) \leqslant \exp(\beta|A_{N,\omega}|(1-\varepsilon)) \overset{N\geqslant N_0(\omega)}{\leqslant} \exp\left(N\frac{\beta}{2}\left(1-\frac{\varepsilon}{2}\right)\right), \tag{A.8}$$

for some $N_0(\omega)$ which is $\mathbb{P}(\mathrm{d}\omega)$-a.s. finite. We can therefore focus on $Z^{\beta,0}_{N,\omega}(\Omega_{N,\omega,\varepsilon})$, which is bounded above by $\exp(\beta|A_{N,\omega}|)\mathbf{P}(\Omega_{N,\omega,\varepsilon})$, and it is thus sufficient to show that

$$\limsup_{\varepsilon\searrow 0}\limsup_{N\to\infty}\frac{1}{N}\log\mathbf{P}(\Omega_{N,\omega,\varepsilon}) \leqslant Q, \tag{A.9}$$

to conclude.

In order to establish (A.9), we introduce a *coarse graining* length $L \in \mathbb{N}$ (L is sent to ∞ in the end, that is, after $N \to \infty$ and $\varepsilon\searrow 0$, so that, in particular, εL can be chosen arbitrarily small and we assume below that $\varepsilon L \ll 1$). We assume that $N/L \in \mathbb{N}$ and we break $\{1, \ldots, N\}$ into N/L non-overlapping blocks $\{B_j\}_{j=1,\ldots,N/L}$ of length L. For every realization of the disorder ω, we decompose the event $\Omega_{N,\omega,\varepsilon}$ into the disjoint union of the events $\Omega_{N,\omega,\varepsilon,\underline{v}}$, $\underline{v} \in \{0,1\}^{N/L}$, defined by the property that if $S \in \Omega_{N,\omega,\varepsilon,\underline{v}}$ then $B_j \cap \{n : S_n = 0\} = B_j \cap A_{N,\omega}$ if and only if $v_j = 0$. In short, there is a mismatch in the block B_j with respect to the energetically optimal contact configuration if and only if $v_j = 1$ (the mismatch can be on a single site or on several sites). Note that there cannot be more than εN blocks containing a mismatch, that is,

if $|\underline{v}| := |\{i : v_i = 1\}| > \varepsilon N$ the event $\Omega_{N,\omega,\varepsilon,\underline{v}}$ is empty. Since $\varepsilon L \ll 1$, only a small fraction of the blocks contains mismatches. We therefore have

$$\mathbf{P}(\Omega_{N,\omega,\varepsilon}) = \sum_{|\underline{v}|\leqslant\varepsilon N} \mathbf{P}(\Omega_{N,\omega,\varepsilon,\underline{v}}) \leqslant \exp(c(\varepsilon L)N/L) \max_{|\underline{v}|\leqslant\varepsilon N} \mathbf{P}(\Omega_{N,\omega,\varepsilon,\underline{v}}), \tag{A.10}$$

where $c(x) \overset{x\searrow 0}{\longrightarrow} 0$ comes from estimating the cardinality of $\{\underline{v} : |\underline{v}| \leqslant \varepsilon N\}$.

We are left with estimating $\mathbf{P}(\Omega_{N,\omega,\varepsilon,\underline{v}})$ uniformly in $\underline{v}$. For this we introduce the random variable $\kappa_j = \inf\{n \in B_j : \omega_n = 1\}$ ($\kappa_j = \infty$ if $\omega_n = 0$ for every $n \in B_j$) and for every j such that $\kappa_j < \infty$ the event

$$E^j_{L,\omega} := \{S : B_j \cap \{n : S_n = 0\} \cap [\kappa_j, N] = A_{N,\omega} \cap B_j \cap [\kappa_j, N]\}, \tag{A.11}$$

that is simply the event that there is no mismatch in B_j from step κ_j onward. If $\kappa_j = \infty$ then $E^j_{L,\omega}$ is just the set of all possible polymer trajectories, without any restriction. Finally, we define

$$Y_j(\omega) := \begin{cases} \mathbf{P}\big(E^j_{L,\omega} \big| S_{\kappa_j} = 0\big) & \text{if} \quad \kappa_j < \infty \\ 1 & \text{otherwise,} \end{cases} \tag{A.12}$$

where, if A and B are two events, $\mathbf{P}(A|B)$ denotes as usual the probability of A conditioned to B. Note that $\{Y_j\}_j$ is a sequence of IID random variables and that, by the renewal theorem (very much like in the proof of the lower bound: we are essentially evaluating the same quantity), $(1/L)\log Y_1(\omega) \overset{L\to\infty}{\longrightarrow} Q$, $\mathbb{P}(\mathrm{d}\omega)$-a.s. Therefore, for every $\delta \in (0, 1)$ we can find L_0 such that for $L \geqslant L_0$,

$$\mathbb{P}\left(Y_1(\omega) \leqslant \exp((Q + \delta)L)\right) \geqslant 1 - \frac{\delta}{2}. \tag{A.13}$$

Now note that

$$\mathbf{P}(\Omega_{N,\omega,\varepsilon,\underline{v}}) \leqslant \mathbf{P}\big(\cap_{j:v_j=0} E^j_{L,\omega}\big), \tag{A.14}$$

and if we set $k := |\{i : v_i = 0\}|$ and $\{i : v_i = 0\} = \{i_1, \ldots, i_k\}$ ($\{i_l\}_{l=1,\ldots,k}$ increasing) we have that if $\kappa_{i_k} < \infty$

$$\mathbf{P}\big(\cap_{j:v_j=0} E^j_{L,\omega}\big) = \mathbf{P}\big(\cap_{l=1}^{k-1} E^{i_l}_{L,\omega} \cap \big\{S_{\kappa_{i_k}} = 0\big\} \cap E^{i_k}_{L,\omega}\big) \leqslant \mathbf{P}\big(\cap_{l=1}^{k-1} E^{i_l}_{L,\omega}\big) Y_{i_k}(\omega), \tag{A.15}$$

where we have used the Markov property of S and in the last step we have neglected the event $\big\{S_{\kappa_{i_k}} = 0\big\}$. With our definition of $Y_j(\omega)$, the factorization inequality in (A.15) actually holds also for $\kappa_{i_k} = \infty$, so that by iterating we obtain

$$\mathbf{P}\big(\cap_{j:v_j=0} E^j_{L,\omega}\big) \leqslant \prod_{j:v_j=0} Y_j(\omega). \tag{A.16}$$

By putting equations (A.13) to (A.16) together, by applying the strong law of large numbers and by exploiting the fact that more than a fraction $(1 - \varepsilon L)$ of the N/L blocks is free of mismatches, we obtain that

$$\mathbf{P}(\Omega_{N,\omega,\varepsilon,\underline{v}}) \leqslant \exp\left((1 - \delta)(Q + \delta)L(1 - 2\varepsilon L)\frac{N}{L}\right), \tag{A.17}$$

for $N \geqslant N_0(\omega)$, with $N_0(\omega)$ a random value that is $\mathbb{P}(\mathrm{d}\omega)$-a.s. finite.

Overall we have therefore established that $\mathbb{P}(\mathrm{d}\omega)$-a.s.

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbf{P}(\Omega_{N,\omega,\varepsilon}) \leqslant \frac{c(\varepsilon L)}{L} + (1 - \delta)(Q + \delta)(1 - 2\varepsilon L) \overset{\varepsilon\searrow 0}{\longrightarrow} (1 - \delta)(Q + \delta). \tag{A.18}$$

Since δ can be chosen arbitrarily small, the proof of the upper bound is complete. □

A.2. Estimates in the periodic case

We prove here theorem 3.1. The first result, that is the small temperature expansion in (3.1), just comes from evaluating the ground state energy. We omit the details since they are substantially easier than those needed for the analogous result in the disordered set-up (cf lemma 4.3 and appendix A.1). We also point out that such a result follows directly from the semi-explicit solution available for periodic models [4, 5] that we now outline since we exploit it in order to establish the high temperature expansion (3.2).

The free energy of periodic models can be expressed by first introducing the (Abelian) group $\mathbb{S} := \mathbb{Z}/(\mathrm{T}(\omega)\mathbb{Z})$, that is $\{1, \dots, \mathrm{T}(\omega)\}$ with periodic boundaries. With abuse of notation an element α of $\mathbb{S}$ is going to be identified with a point in $\{1, \dots, \mathrm{T}(\omega)\}$, so by $n \in \alpha$ we mean $n = k\mathrm{T}(\omega) + \alpha$ for some $k \in \mathbb{Z}$. For $b \geqslant 0$, we set

$$K_\alpha(b) := \sum_{n\in\alpha} K(n) \exp(-bn), \tag{A.19}$$

and in turn for α and $\gamma \in \mathbb{S}$

$$A_{\gamma,\alpha}(b, \beta) := K_{\alpha-\gamma}(b) \exp(\beta\omega_\alpha). \tag{A.20}$$

By the Perron–Frobenius theory on matrices with positive entries, the $\mathrm{T}(\omega) \times \mathrm{T}(\omega)$–matrix $A(b, \beta)$ has a maximal positive eigenvalue, often called the *spectral radius* of $A(b, \beta)$, that we denote by $\lambda_\omega(b, \beta)$. By standard arguments, one shows that $\lambda_\omega(\cdot, \beta)$ is decreasing and smooth. In [4, 5] it is shown that the (ω-dependent) free energy is 0 if $\lambda_\omega(0, \beta) \leqslant 1$. If instead $\lambda_\omega(0, \beta) > 1$ then there exists a unique solution $b > 0$ to the equation $\lambda_\omega(b, \beta) = 1$, and such a b is precisely the free energy $\mathrm{F}(\beta)$.

Let us now expand $A(b, \beta)$ for small values of b and β:

$$A_{\gamma,\alpha}(b, \beta) = A_{\gamma,\alpha}(0, 0)\left(1 + \beta\omega_\alpha + \frac{1}{2}\beta^2 + o(\beta^2)\right) - \frac{\sqrt{2p}}{\mathrm{T}(\omega)}(b^{1/2} + o(b^{1/2})), \tag{A.21}$$

where the last term follows from the Riemann sum approximation procedure,

$$\begin{aligned}\sum_{n\in\alpha}(1 - \exp(-bn))K(n) &= b^{1/2}(\sqrt{p/(2\pi)} + o(1))b \sum_{n\in\alpha} \frac{(1 - \exp(-bn))}{(bn)^{3/2}} \\ &= \frac{1}{\mathrm{T}(\omega)} b^{1/2}(\sqrt{p/(2\pi)} + o(1)) \int_0^\infty \frac{(1 - \exp(-z))}{z^{3/2}}\, \mathrm{d}z,\end{aligned} \tag{A.22}$$

and by the fact that the integral in the last term is equal to $2\sqrt{\pi}$. We now use the fact that the maximal eigenvalue $\lambda(A + \varepsilon B)$, A as matrix with positive terms and ε small, can be written up to $O(\varepsilon^2)$ terms as $\lambda(A) + \varepsilon u \cdot Bv$, with u and v respectively as the right and left eigenvectors of A with the eigenvalue $\lambda(A)$, normalized by setting $\sum_\alpha v_\alpha = 1$ and $\sum_\alpha u_\alpha v_\alpha = 1$. In our case $A = A(0, 0)$ turns out to be bi-stochastic, so $\lambda(A)$, before denoted as $\lambda_\omega(0, 0)$, is equal to 1 and $v_\alpha = 1/\mathrm{T}(\omega)$, as well as $u_\alpha = 1$ for every α. This leads to the expansion

$$\lambda(b, \beta) = 1 + \beta\left(\frac{1}{\mathrm{T}(\omega)} \sum_{n=1}^{\mathrm{T}(\omega)} \omega_n\right) + \frac{1}{2}\beta^2 - \sqrt{2p}b^{1/2} + r(b, \beta), \tag{A.23}$$

where $r(b, \beta) = O(\beta^2) + o(\sqrt{b})$ if $\sum_{n=1}^{\mathrm{T}(\omega)} \omega_n \neq 0$ and $r(b, \beta) = o(\beta^2) + o(\sqrt{b})$ otherwise. Therefore, the existence of a (unique) solution $b = \mathrm{F}(\beta)$ to $\lambda_\omega(b, \beta) = 1$ for β small requires $\sum_{n=1}^{\mathrm{T}(\omega)} \omega_n \geqslant 0$ and

$$\mathrm{F}(\beta) = \begin{cases} \frac{\beta^2}{2p} \frac{1}{\mathrm{T}(\omega)} \sum_{n=1}^{\mathrm{T}(\omega)} \omega_n + o(\beta^2) & \text{if} \quad \sum_{n=1}^{\mathrm{T}(\omega)} \omega_n > 0, \\ \frac{\beta^4}{8p} + o(\beta^4) & \text{if} \quad \sum_{n=1}^{\mathrm{T}(\omega)} \omega_n = 0. \end{cases} \tag{A.24}$$

Armed with these asymptotic behaviours, (3.2) follows from $\mathrm{G}^{-1}(y) = \sqrt{2y(1+o(1))/p}$ $(y \searrow 0)$.

What happens when $\sum_{n=1}^{\mathrm{T}(\omega)} \omega_n < 0$ is that $\lambda_\omega(0,\beta)$ is smaller than 1 for small β (see (A.23)). If we set $\beta_0 := \sup\{\beta : \lambda_\omega(0,\beta) < 0\}$ (note that $\beta < \infty$ unless $\omega_n = -1$ for every n, as one can see from the large β expansion), then one readily sees that $f_c(\beta) > 0$ for $\beta > \beta_0$ and $f_c(\beta) = 0$ otherwise.

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